

A Greedy Approximation for Minimum Cardinality Multiple Quasi-submodular Cover with Applications

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Abstract

Quasi-submodularity is a unified measurement to characterize submodularity or approximate submodularity of a set function. In this paper, we consider a variant of Minimum Cardinality Cover named Minimum Cardinality Multiple Quasi-submodular Cover, which requires to find the smallest subset of given ground set such that multiple quasi-submodular functions reach a maximum. We design and analyze a greedy approximation algorithm for Minimum Cardinality Multiple Quasi-submodular Cover. As some applications of our results, we achieve a $O(\ln n)$ -approximation algorithm for Minimum Resolving Restrained Dominating Set, and a $O(\ln \delta)$ -approximation algorithm for Minimum Dominating Set in multiplex networks, where n is the node number and δ is the maximum node degree of the input graph.

Index Terms— Minimum cardinality cover, Submodular, Approximation algorithm, Dominating set

1 Introduction

Minimum Cardinality Cover(MCC for short) is a well-known NP-hard combinatorial optimization problem. Given a ground set U and a set function $f:2^U \rightarrow \mathbb{R}^+$ which is normalized ($f(\emptyset) = 0$) and non-decreasing ($f(T) \geq f(R)$ whenever $T \subseteq R \subseteq U$), MCC is asked to find a smallest subset $S \subseteq U$ such that $f(S) = f(U)$, more formally,

$$\min_{S \subseteq U} \{|S| : f(S) = f(U)\}.$$

For any pair of $T, R \subseteq U$, the marginal gains of set R in T is denoted by $\Delta_R f(T) = f(R \cup T) - f(T)$. Specially, denote by $\Delta_i f(T) = f(T \cup \{i\}) - f(T)$ the marginal gains of singleton set $\{i\}$ in T . The classical greedy algorithm(CGA for short) is a basic method for MCC: starts from an empty set; iteratively adds to the current solution set one element with the maximum marginal gains until potential function $f(\cdot)$ reaches the maximum.

Especially, suppose potential function $f(\cdot)$ satisfies submodularity or approximate submodularity, then algorithm CGA enjoys some provable approximation guarantees for

Algorithm 1 CGA

Input: A ground set U and potential function $f:2^U \rightarrow \mathbb{R}^+$

Output: A subset $S \subseteq U$ such that $f(S) = f(U)$

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1:  $S \leftarrow \emptyset$ ;
2: while  $\exists u \in U$  such that  $\Delta_u f(S) > 0$  do
3:    $u \leftarrow \arg \max_{u \in U \setminus S} \Delta_u f(S)$ ;
4:    $S \leftarrow S \cup \{u\}$ ;
5: end while
6: return  $S$ .
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these variants of MCC. Now we introduce the submodularity and two approximate submodularities.

(i): Submodularity [1, 2, 3]. The submodularity of $f(\cdot)$ is that the marginal gains of joining an element to a set is not less than that of joining an element to a superset. More formally, for any $T \subseteq R \subseteq U$ and any $i \in U \setminus R$, $f(\cdot)$ satisfies

$$\Delta_i f(T) \geq \Delta_i f(R).$$

(ii): Submodularity Ratio [4, 5]. The submodularity ratio is a approximate submodularity which is used to characterize how close a set function is to be submodular. More formally, the submodularity ratio of $f(\cdot)$ is the largest scalar $\gamma \in (0,1]$ such that for any $T \subseteq R \subseteq U$ and any $i \in U \setminus R$, $f(\cdot)$ satisfies

$$\Delta_i f(T) \geq \gamma \times \Delta_i f(R).$$

(iii): Submodularity Gap [5, 6]. The submodularity gap is another approximate submodularity which is also used to characterize how close a set function is to be submodular. More formally, the submodularity gap of f is the smallest scalar $\theta \in [0,+\infty)$ such that for any $T \subseteq R \subseteq U$ and any $i \in U \setminus R$, $f(\cdot)$ satisfies

$$\Delta_i f(T) \geq \Delta_i f(R) - \theta.$$

Submodularity and approximate submodularity play an important role in combinatorial optimization, especially the Minimum Cardinality Cover. However, some combinatorial optimization problems require multiple submodular or approximate submodular functions reach a maximum. These problems are usually NP-hard, which means it is unlikely to be solved precisely in polynomial time unless $P = NP$. Greedy approximation algorithm might be very popular to solve the problem, because it enjoy low time complexity and thus can

be suitable for big data, and particularly because it can give solutions with guaranteed quality, which are close to the best that could theoretically be obtained. The contributions of this paper are as follows.

- Define a measurement which is called quasi-submodularity to uniformly characterize submodularity and some approximate submodularities.
- Design and analyze a greedy approximation algorithm for Minimum Cardinality Multiple Quasi-submodular Cover. It can be proved that our approximation ratio slightly generalizes some known ones.
- Achieve a $O(\ln n)$ -approximation algorithm for the minimum resolving restrained dominating sets problem, and a $O(\ln \delta)$ -approximation algorithm for the minimum dominating sets problem in multiplex networks, where n is the node number and δ is the maximum node degree of the input graph.

2 Preliminaries

First, we give a measurement which is called quasi-submodularity to unified characterize submodularity and some approximate submodularities.

Definition 1. (*Quasi-submodularity*). Given a ground set U and a set function $f: 2^U \rightarrow \mathbb{R}^+$, the quasi-submodularity is binary parameters (λ, μ) such that for any $T \subseteq R \subseteq U$ and any $i \in U \setminus R$,

$$\Delta_i f(T) \geq \lambda \times \Delta_i f(R) - \mu,$$

where $\lambda \in (0, 1]$ and $\mu \in [0, +\infty)$.

For Convenience, henceforth we say that a function $f(\cdot)$ is (λ, μ) -submodular if its quasi-submodularity is (λ, μ) . It is easy to see that if $f(\cdot)$ satisfies (λ, μ) -submodularity, then $f(\cdot)$ satisfies (λ', μ') -submodularity for any $\lambda' \in (0, \lambda]$ and $\mu' \in [\mu, +\infty)$.

Remarks:

- (i) $(1, 0)$ -submodularity is the classic submodularity,
- (ii) $(\lambda, 0)$ -submodularity is called submodularity ratio of $f(\cdot)$ if and only if λ is the largest scalar such that $f(\cdot)$ satisfies $(\lambda, 0)$ -submodularity,
- (iii) $(1, \mu)$ -submodularity is called submodularity gap of $f(\cdot)$ if and only if μ is the smallest scalar such that $f(\cdot)$ satisfies $(1, \mu)$ -submodularity.

Next, we consider a variant of Minimum Cardinality Cover. The variant requires multiple submodular or approximate submodular functions reach a desirable fraction.

Definition 2. (*Minimum Cardinality Multiple Quasi-Submodular Cover*). Given a ground set U and multiple set functions $f_1, \dots, f_k: 2^U \rightarrow \mathbb{R}^+$, where f_i is normalized, non-decreasing and (λ_i, μ_i) -submodular for any $i \in \{1, 2, \dots, k\}$, Minimum Cardinality Multiple Quasi-Submodular Cover is asked to find a smallest subset $S \subseteq U$ such that $f_i(S) = f_i(U)$ for any $i \in \{1, 2, \dots, k\}$, more formally,

$$\min_{S \subseteq U} \{|S| : f_i(S) = f_i(U), i \in \{1, 2, \dots, k\}\}.$$

Since that Minimum Cardinality Multiple Quasi-submodular Cover generalizes the problem of covering multiple submodular constraints [7] if (λ_k, μ_k) -submodularity is submodularity for any $i \in \{1, 2, \dots, k\}$, then Minimum Cardinality Multiple Quasi-submodular Cover is NP-hard, which means it is unlikely to be solved precisely in polynomial time unless $P = NP$. Our aim is to give a greedy approximation algorithm with polynomial time.

3 Greedy approximation algorithm

For Minimum Cardinality Multiple Quasi-submodular Cover, we give a solution with guaranteed quality by considering a potential function $f: 2^U \rightarrow \mathbb{R}^+$ for any $S \subseteq U$, where

$$f(S) = \sum_{i=1}^k f_i(S).$$

It is easy to see $f(\cdot)$ is normalized and non-decreasing due to $f_i(\cdot)$ is normalized and non-decreasing for any $i \in \{1, 2, \dots, k\}$. Further, we give some properties of $f(S)$ for any $S \subseteq U$.

Lemma 1. *The potential function $f(\cdot)$ satisfies*

- (i) (λ, μ) -submodularity, where $\lambda = \min\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and $\mu = \sum_{i=1}^k \mu_i$,
- (ii) $f(S) = f(U)$ if and only if S is a solution for Minimum Cardinality Multiple Quasi-submodular Cover, and
- (iii) $f(S) < f(U)$ if and only if there exists a element $u \in U - S$ and $i \in \{1, 2, \dots, k\}$ such that $\Delta_u f_i(S) > 0$.

Proof. (i) Since $f_i(\cdot)$ is (λ_i, μ_i) -submodular for any $i \in \{1, 2, \dots, k\}$, we have for any $A \subseteq B \subseteq U$ and any $u \in U - B$

$$\begin{aligned} \Delta_u f(A) &= \sum_{i=1}^k \Delta_u f_i(A) \\ &\geq \sum_{i=1}^k (\lambda_i \times \Delta_u f_i(B) - \mu_i) \\ &> \lambda \times \left(\sum_{i=1}^k \Delta_u f_i(B) \right) - \mu \\ &= \lambda \times \Delta_u f(B) - \mu, \end{aligned}$$

where $\lambda = \min\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and $\mu = \sum_{i=1}^k \mu_i$.

(ii) $f(S) = f(U)$, namely, $f_i(S) = f_i(U)$ for $i \in \{1, 2, \dots, k\}$ if and only if S is a solution for Minimum Cardinality Multiple Quasi-submodular Cover.

(iii) $f(S) < f(U)$, namely, there exist $u \in U - S$ such that $\Delta_u f(S) > 0$, which is equivalent to the existence of $i \in \{1, 2, \dots, k\}$ and $u \in U - S$ such that $\Delta_u f_i(S) > 0$. $\square$

Next, we prove that algorithm CGA yields a approximation solution for Minimum Cardinality Multiple Quasi-submodular Cover.

Theorem 1. *If $f(\cdot)$ is integer-valued, there exists a $(1 + \lambda\mu + \frac{1}{\mu} \ln \alpha)$ -approximation solution yielded by CGA for finding a subset $S \subseteq U$ such that $f(S) = f(U)$, where $\alpha = \max_{u \in U} f(\{u\})$.*

Proof. Suppose that the greedy solution yielded by CGA is $S = \{s_1, s_2, \dots, s_g\}$, where g is the cardinality of the greedy solution and s_i is the element selected in the i th iteration of the algorithm for $i = 1, 2, \dots, g$. And suppose $S_i = \{s_1, s_2, \dots, s_i\}$ for $i = 1, 2, \dots, g$ and $S_0 = \emptyset$. Similarly, we let $O = \{o_1, o_2, \dots, o_{opt}\}$ be an optimal solution where opt is its cardinality, and let $O_j = \{o_1, o_2, \dots, o_j\}$ for $j = 1, 2, \dots, opt$ and $O_0 = \emptyset$.

From greedy rule, we have

$$s_i = \arg \max_{u \in U \setminus S_{i-1}} \Delta_u f(S_{i-1}),$$

namely, for any $i \in \{0, 1, \dots, g-1\}$ and any $j \in \{1, 2, \dots, opt\}$,

$$f(S_{i+1}) - f(S_i) = \Delta_{s_{i+1}} f(S_i) \geq \Delta_{o_j} f(S_i),$$

and thus we have

$$f(S_{i+1}) - f(S_i) \geq \frac{1}{opt} \sum_{j=1}^{opt} \Delta_{o_j} f(S_i).$$

Additionally, for any $i \in \{0, 1, \dots, g\}$, we have

$$f(U) - f(S_i) = f(S_i \cup O_{opt}) - f(S_i) = \sum_{j=1}^{opt} \Delta_{o_j} f(S_i \cup O_{j-1}).$$

From lemma 1(i), we have

$$\Delta_{o_j} f(S_i) \geq \lambda \times \Delta_{o_j} f(S_i \cup O_{j-1}) - \mu,$$

where $\lambda = \min\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and $\mu = \sum_{i=1}^k \mu_i$. We immediately establish the following recursive inequality for any $i \in \{0, 1, \dots, g-1\}$

$$f(S_{i+1}) - f(S_i) \geq \frac{\lambda}{opt} (f(U) - f(S_i) - \mu \times opt),$$

which implies

$$\lambda \times \left(\frac{f(U)}{opt} - \mu \right) \leq f(\{s_1\}) - f(\emptyset) \leq \Delta_{s_1} f(\emptyset) = \max_{u \in U} f(\{u\}).$$

Let $c_{i-1} = \lambda \times (f(U) - f(S_{i-1}) - \mu \times opt)$ and $\alpha = \max_{u \in U} f(\{u\})$, we have

$$c_g = -\lambda \times \mu \times opt$$

and

$$c_0 = \lambda \times (f(U) - \mu \times opt) = \lambda \times \left(\frac{f(U)}{opt} - \mu \right) \times opt \leq \alpha \times opt.$$

For any $i \in \{1, 2, \dots, g\}$, another recursive inequality is established as

$$c_{i-1} - c_i = \lambda \times \Delta_{s_i} f(S_{i-1}) \geq \frac{\lambda \times c_{i-1}}{opt},$$

which implies

$$c_i \leq c_0 \left(1 - \frac{\lambda}{opt}\right)^i \leq c_0 e^{-\frac{\lambda \times i}{opt}}.$$

Note that since c_i is decreasing with respect to i , there exists $x \in \{1, 2, \dots, g-1\}$ such that for any $\eta \in (0, \alpha)$ we have $c_x = (\eta - \lambda \times \mu) \times opt$. From c_x to c_g , c_x will reduce from $(\eta - \lambda \times \mu) \times opt$ to $-\lambda \times \mu \times opt$ with $g - x$ elements are selected by CGA. Otherwise, by assuming that $f(\cdot)$ is integer-valued, c_x reduces 1 at least when any element in $U \setminus S_x$ is selected. Therefore, we have $g \leq \eta \times opt + x$ from

$$opt \times (\eta - \lambda \times \mu) - (g - x) \geq -\lambda \times \mu \times opt.$$

By $c_x \leq c_0 e^{-\frac{\lambda \times x}{opt}}$, the relationship between x and opt can be obtained as

$$\begin{aligned} x &\leq \frac{opt}{\lambda} \times \ln \frac{c_0}{c_x} \\ &\leq \frac{opt}{\lambda} \times \ln \frac{\alpha \times opt}{opt \times (\eta - \lambda \times \mu)} \\ &\leq \frac{opt}{\lambda} \times \ln \frac{\alpha}{\eta - \lambda \times \mu}. \end{aligned}$$

Hence,

$$g \leq \eta \times opt + x \leq opt \times \left(\eta + \frac{1}{\lambda} \ln \frac{\alpha}{\eta - \lambda \times \mu} \right).$$

In order to get the lower bound of approximation ratio, we consider function $p(\eta) = \eta + \frac{1}{\lambda} \ln \frac{\alpha}{\eta - \lambda \times \mu}$ is minimal at $\eta = 1 + \lambda \times \mu$. Since $\eta \in (0, \alpha)$, we can simplify the approximation ratio to $1 + \lambda\mu + \frac{1}{\lambda} \ln \alpha$. $\square$

We remark that Theorem 1 slightly generalizes some known results. For Minimum Cardinality Submodular Cover, it is easy to obtain from Theorem 1 the approximation ratio $1 + \ln \alpha$ which is almost the same as $H(\alpha)$. Theorem 1 can also directly derive an approximation ratio $1 + \frac{1}{\lambda} \ln \alpha$ for Minimum Cardinality Ratio-submodular Cover if $(\lambda, 0)$ -submodularity is submodularity ratio of $f(\cdot)$, and another approximation ratio $1 + \mu + \ln \alpha$ for Minimum Cardinality Gap-submodular Cover if $(1, \mu)$ -submodularity is submodularity gap of $f(\cdot)$.

4 Application

In the section, we would like to generalize our results to Minimum Resolving Restrained Dominating Set and Minimum Dominating Set in Multiplex Networks.

4.1 Minimum Resolving Restrained Dominating Set

Given a connected graph $G = \langle V, E \rangle$ with node set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E . The order of G is given by $n = |V|$ and its size by $m = |E|$. For any $u, v \in V$, we denote by $N(v)$ the set of all neighbors of v , by $dis(u, v)$ the shortest distance of u, v in G . Next, we introduce some concepts

including dominating set, resolving set and Minimum Resolving Restrained Dominating Set.

For any $D \subseteq V$ and any node $u \in V$, u is dominated by D if there exists a node $v \in D$ such that $v \in N(u)$. Any subset $D \subseteq V$ is called a dominating set if and only if any node in $V - D$ can be dominated by at least one node in D . In order to formally describe the dominating set, we define a potential function $f_1 : 2^V \rightarrow \mathbb{Z}^+$ for any $D \subseteq V$, where

$$f_1(D) = |\{v|v \in V - D, N(v) \cap D \neq \emptyset\}| + |D|. \quad (1)$$

Obviously, $f_1(\emptyset) = 0$, $f_1(D) < n$ if and only if the existence of $v \in V - D$ such that $N(v) \cap D = \emptyset$, namely, any subset $D \subseteq V$ is called a dominating set if and only if $f_1(D) = f_1(V) = n$.

For any $D \subseteq V$ and any two distinct nodes $u, w \in V$, u and w are resolved by D if there exists a node $v \in D$ such that $dis(v, u) \neq dis(v, w)$. Any subset $D \subseteq V$ is called a resolving set [8] if and only if any two distinct nodes in V can be resolved by D . In order to formally describe the resolving set, we construct a set $V' = \{(v_i, v_j)|v_i \in V, v_j \in V, i < j\}$ in polynomial time, let $C((v_i, v_j)) = \{v|v \in V, dis(v, v_i) \neq dis(v, v_j)\}$ for any $(v_i, v_j) \in V'$, and define a potential function $f_2 : 2^V \rightarrow \mathbb{Z}^+$ for any $D \subseteq V$, where

$$f_2(D) = |\{(v_i, v_j)|v_i, v_j \in V', C((v_i, v_j)) \cap D \neq \emptyset\}|. \quad (2)$$

Obviously, $f_2(\emptyset) = 0$, $f_2(D) < \frac{n(n+1)}{2}$ if and only if the existence of $(v_i, v_j) \in V'$ such that $C((v_i, v_j)) \cap D = \emptyset$, namely, any subset $D \subseteq V$ is called a resolving set if and only if $f_2(D) = f_2(V) = \frac{n(n+1)}{2}$. It's worth recalling that the resolving set with smallest cardinality is called *metric dimension* [9] in G . Finding the *metric dimension* in a general graph is NP-hard [10], which means it is unlikely to be solved precisely in polynomial time unless $P = NP$. At present, there are two greedy approximation algorithms [8, 11] for finding *metric dimension*, in which the greedy rule of Khuller-Greedy [8] with $(1 + 2 \ln n)$ -approximation is equivalent to selecting $v \in V - D$ with $\max_{v \in V - D} \Delta_v f_2(D)$ in each iteration.

The Minimum Resolving Restrained Dominating Set [12] is asked to find a smallest subset $D \subseteq V$ with smallest cardinality such that D is a dominating set and resolving set. More formally,

$$\min_{D \subseteq V} \{|D| : f_1(D) = f_1(V), f_2(D) = f_2(V)\}.$$

To the best of our knowledge, there is no approximation algorithm to solve Minimum Resolving Restrained Dominating Set so far. Our aim is to propose an approximation algorithm by considering a potential function $f : 2^V \rightarrow \mathbb{Z}^+$ for any $D \subseteq V$, where

$$f(D) = f_1(D) + f_2(D). \quad (3)$$

Let us show some properties of $f(D)$.

Lemma 2. *The potential function $f(\cdot)$ satisfies*

- (i) $f(\cdot)$ is normalized, non-decreasing and $(1, 0)$ -submodular;
- (ii) $f(D) = f(V)$ if and only if D is a solution for Minimum

Resolving Restrained Dominating Set, and

- (iii) $f(D) < f(V)$ if and only if there exists a node $v \in V - D$ such that $\Delta_v f(D) > 0$.

Proof. (i) From (1) and (2), $f(\emptyset) = 0$ is clear. It is easy to see that $f(\cdot)$ is non-decreasing due to $|N(v) \cap D|$ and $|C((v_i, v_j)) \cap D|$ are non-decreasing with respect to D for any $v \in V - D$ and any $(v_i, v_j) \in V'$. For any $A \subseteq B \subseteq V$ and any $x \in V - B$, we have

$$\begin{aligned} \Delta_x f_1(A) &= f_1(A \cup \{x\}) - f_1(A) \\ &= |N[x] - A - \{v|v \in V, N(v) \cap A \neq \emptyset\}| \\ &\geq |N[x] - B - \{v|v \in V, N(v) \cap B \neq \emptyset\}| \\ &= \Delta_x f_1(B) \end{aligned}$$

and

$$\begin{aligned} \Delta_x f_2(A) &= f_2(A \cup \{x\}) - f_2(A) \\ &= |C(x) - \{(v_i, v_j)|v_i, v_j \in V', C((v_i, v_j)) \cap A = \emptyset\}| \\ &\geq |C(x) - \{(v_i, v_j)|v_i, v_j \in V', C((v_i, v_j)) \cap B = \emptyset\}| \\ &= \Delta_x f_2(B), \end{aligned}$$

hence,

$$\begin{aligned} \Delta_x f(A) &= \Delta_x f_1(A) + \Delta_x f_2(A) \\ &\geq \Delta_x f_1(B) + \Delta_x f_2(B) \\ &= \Delta_x f(B). \end{aligned}$$

(ii) $f(D) = f(V)$, namely, $f_1(D) = f_1(V) = n$ and $f_2(D) = f_2(V) = \frac{n(n+1)}{2}$ if and only if $N(v) \cap D \neq \emptyset$ for every $v \in V - D$ and $C((v_i, v_j)) \cap D \neq \emptyset$ for every $(v_i, v_j) \in V'$ if and only if D is a solution for Minimum Resolving Restrained Dominating Set.

(iii) $f(D) < f(V)$, namely, $f_1(D) < n$ or $f_2(D) < \frac{n(n+1)}{2}$ if and only if the existence of $v \in V - D$ such that $N(v) \cap D = \emptyset$ or $(v_i, v_j) \in V'$ such that $C((v_i, v_j)) \cap D = \emptyset$, which is equivalent to the existence of $v \in V - D$ such that $\Delta_v f(D) > 0$. $\square$

Based on Lemma 2 and Theorem 1, it is easy to know algorithm CGA yields a $(1 + 2 \ln(n + 1))$ -approximation solution for Minimum Resolving Restrained Dominating Set due to

$$\begin{aligned} \max_{v \in V} f(\{v\}) &\leq \max_{v \in V} (f_1(\{v\}) + f_2(\{v\})) \\ &= \max_{v \in V} (|N[v]|) + \max_{v \in V} (|C(v)|) \\ &< n + n^2 < (n + 1)^2. \end{aligned}$$

4.2 Minimum Dominating Set in Multiplex Networks

Considering a multiplex network which is formed by k different layers, i.e., $G = (G_1, \dots, G_k)$, where every layer $G_i = (V, E_i)$ has a same set V of n nodes and a distinct set of edges E_i . We denote by $N_i(v)$ the set of all neighbors of v in $G_i = (V, E_i)$. In order to formally describe the dominating set

in $G_i = (V, E_i)$, we define a potential function $f_i : 2^V \rightarrow \mathbb{Z}^+$ for any $D \subseteq V$ and any $i \in \{1, 2, \dots, k\}$, where

$$f_i(D) = |\{v|v \in V - D, N_i(v) \cap D \neq \emptyset\}| + |D|. \quad (4)$$

Obviously, $f_i(\emptyset) = 0$, $f_i(D) < n$ if and only if the existence of $v \in V - D$ such that $N_i(v) \cap D = \emptyset$, namely, any subset $D \subseteq V$ is called a dominating set in $G_i = (V, E_i)$ if and only if $f_i(D) = f_i(V) = n$. The Minimum Dominating Set in Multiplex Networks [13] is asked to find a smallest subset $D \subseteq V$ with smallest cardinality such that D is a dominating set in every layer $G_i = (V, E_i)$. More formally,

$$\min_{D \subseteq V} \{|D| : f_i(D) = f_i(V), i \in \{1, 2, \dots, k\}\}.$$

Even with $k = 1$, Minimum Dominating Set in Multiplex Networks generalizes the well-known Minimum Dominating Set. Minimum Dominating Set in Multiplex Networks is NP-hard problem due to it can also be easily reduced to Minimum Dominating Set. Some practical applications such as Monitoring Epidemic in Multiplex Network [14], Early Detecting and Controlling Epidemic in Multiplex Network [15] and Constructing Extractive Text Summarization [16] are usually modeled as Minimum Dominating Set in Multiplex Networks.

Some heuristic algorithms [13, 17] for the Minimum Dominating Set in Multiplex Networks have been proposed. Our aim is to propose an approximation algorithm by considering a potential function $f : 2^V \rightarrow \mathbb{Z}^+$ for any $D \subseteq V$, where

$$f(D) = \sum_{i=1}^k f_i(D). \quad (5)$$

Let us show some properties of $f(D)$.

Lemma 3. *The potential function $f(\cdot)$ satisfies*

- (i) $f(\cdot)$ is normalized, non-decreasing and $(1, 0)$ -submodular;
- (ii) $f(D) = f(V)$ if and only if D is a solution for Minimum Dominating Set in Multiplex Networks, and
- (iii) $f(D) < f(V)$ if and only if there exists a node $v \in V - D$ such that $\Delta_v f(D) > 0$.

Proof. (i) From (4), $f(\emptyset) = 0$ is clear. It is easy to see that $f(\cdot)$ is non-decreasing due to $|N_i(v) \cap D|$ is non-decreasing with respect to D for any $v \in V - D$ and any $i \in \{1, 2, \dots, k\}$. For any $A \subseteq B \subseteq V$ and any $x \in V - B$, we have

$$\begin{aligned} \Delta_x f_i(A) &= f_i(A \cup \{x\}) - f_i(A) \\ &= |N[x] - A - \{v|v \in V, N(v) \cap A \neq \emptyset\}| \\ &\geq |N[x] - B - \{v|v \in V, N(v) \cap B \neq \emptyset\}| \\ &= \Delta_x f_i(B), \end{aligned}$$

hence,

$$\begin{aligned} \Delta_x f(A) &= \sum_{i=1}^k \Delta_x f_i(A) \\ &\geq \sum_{i=1}^k \Delta_x f_i(B) \\ &= \Delta_x f(B). \end{aligned}$$

(ii) $f(D) = f(V)$, namely, $f_i(D) = f_i(V) = n$ for any $i \in \{1, 2, \dots, k\}$ if and only if $N_i(v) \cap D \neq \emptyset$ for every $v \in V - D$ and any $i \in \{1, 2, \dots, k\}$ if and only if D is a solution for Minimum Dominating Set in Multiplex Networks.

(iii) $f(D) < f(V)$, namely, the existence of $i \in \{1, 2, \dots, k\}$ such that $f_i(D) < n$ if and only if the existence of $v \in V - D$ and $i \in \{1, 2, \dots, k\}$ such that $N_i(v) \cap D = \emptyset$, which is equivalent to the existence of $v \in V - D$ such that $\Delta_v f(D) > 0$. $\square$

Based on Lemma 3 and Theorem 1, it is easy to know algorithm CGA yields a $(1 + \ln(k \times \delta + k))$ -approximation solution for Minimum Dominating Set in Multiplex Networks due to

$$\begin{aligned} \max_{v \in V} f(\{v\}) &\leq \sum_{i=1}^k \max_{v \in V} f_i(\{v\}) \\ &= \sum_{i=1}^k \max_{v \in V} (|N_i[v]|) \\ &= \sum_{i=1}^k (\delta_i + 1) \\ &< k \times \delta + k, \end{aligned}$$

where δ_i is the maximum node degree of the graph $G_i = (V, E_i)$ and $\delta = \max\{\delta_1, \delta_2, \dots, \delta_k\}$.

5 Conclusion

In this paper, we have proposed a variant of Minimum Cardinality Cover named Minimum Cardinality Multiple Quasi-submodular Cover and given a greedy approximation algorithm for it by defining a quasi-submodular potential function. As some applications of our results, we directly give $(1 + 2 \ln(n + 1))$ -approximation solution for Minimum Resolving Restrained Dominating Set and a $(1 + \ln(k \times \delta + k))$ -approximation solution for Minimum Dominating Set in Multiplex Networks, where n is node number and δ is the maximum node degree of the input graph.

In the future work, we would like to generalize our method to some other related NP-hard problems to explore their greedy approximation algorithms with better approximation ratio or performance.

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